

A Simple Intrinsic Reduced-Observer for Geodesic Flow

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Abstract

Aghannan and Rouchon proposed a new design method of asymptotic observers for a class of nonlinear mechanical systems: Lagrangian systems with configuration (position) measurements. The observer is based on the Riemannian structure of the configuration manifold endowed with the kinetic energy metric and is intrinsic. They proved local convergence. When the system is conservative, we propose a *globally* convergent intrinsic reduced-observer based on the Jacobi metric. For non-conservative systems the observer can be used as a complement to the one of Aghannan and Rouchon. More generally the reduced-observer provides velocity estimation for geodesic flow with position measurements. Thus it can be (formally) used as a fluid flow soft sensor in the case of a perfect incompressible fluid. When the curvature is negative in all planes the geodesic flow is sensitive to initial conditions. Surprisingly this instability yields faster convergence.

Keywords Riemannian curvature, geodesic flow, non-linear asymptotic observer, Lagrangian mechanical systems, intrinsic equations, contraction, infinite dimensional Lie group, incompressible fluid.

There is no general method to design asymptotic observers for observable non-linear systems, but some specific types of linearities have been tackled in the literature. In particular work has been devoted over the last few years to observer design for systems possessing symmetries. Notably the case of a finite-dimensional group of symmetries acting on the state space [2, 7, 13, 9], and left-invariant dynamics on a Lie group have been considered

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[8]. Symmetries correspond in general to invariance to some changes of units and frame. The invariance to any change of coordinates was raised by [3] who designed an intrinsic observer for a class of non-linear systems: Lagrangian systems with position (configuration) measurements. The aim is to estimate the velocity, independently from any nontrivial choice of coordinates, and of course never differentiate the (noisy) output. The observer was adapted to the specific case of a left-invariant system on a Lie group by [14]. Observer [3] is based on the Riemannian structure of the configuration manifold endowed with the kinetic energy metric. This geometry had already been used in control theory of mechanical systems (see e.g. [11, 10]). The convergence of the observer is local.

According to the Maupertuis principle, the motion of a conservative Lagrangian system is a geodesic flow for the Jacobi metric, intrinsically defined after the kinetic and potential energies, up to a time reparametrization. In this paper we consider the general problem of building a reduced observer to estimate the velocity for geodesic flow on a Riemannian manifold (motion along a geodesic with constant speed). Under some basic assumptions relative to the injectivity radius (also formulated in [3]) we have the following results (theorem 1). If there is an upper bound $A > 0$ on the sectional curvature in all planes, the reduced observer is exponentially convergent (the convergence is global in a sense to define) as long as the gain is bounded from below by a linear function of \sqrt{A} . Unfortunately the higher the gain is the most sensitive to noise the observer is. An even better situation occurs when the sectional curvature is non-positive in all planes: the reduced observer is globally exponentially convergent, without restrictions on the gain. In fact, for a fixed gain, the more negative the curvature is the faster the observer converges. This feature is surprising enough as negative curvature implies exponential divergence between two nearby geodesics, and thus “amplifies” initial errors. This is a major difference with [3] who used additional terms precisely to cancel the effects of (negative) curvature.

For mechanical Lagrangian systems the observer of Aghannan and Rouchon is only locally convergent. In the absence of external forces the reduced observer offers a good alternative, as it is globally convergent. When there are external forces, the reduced-observer can be used as a complement to [3]. The gain must be chosen large enough so that the reduced observer converges before the energy varies significantly. If so, it provides the observer [3] with an initial estimated velocity close to the true one.

The reduced observer is also applied, formally, to a basic velocimetry problem: compute the velocity of a perfect incompressible fluid observing the fluid particles. The principle of least action shows that the motion of an incompressible fluid can be viewed as a geometric flow. We consider the case

of a two-dimensional fluid. As the convergence properties of the observer depend on the sign of the curvature, we will use results and heuristics of Arnol'd [5]. Following them we show theoretically that global convergence could be expected for a large class of trajectories, since the curvature is positive only in a few sections. This latter fact also implies instability of the flow, and Arnol'd gives an interpretation relative to the difficulty of weather's prediction. Note that the problem tackled is nontrivial, as the system is nonlinear, infinite dimensionnal, and possibly sensitive to initial conditions.

In Section 1 we give the general motivations introducing Lagrangian systems on manifolds and Maupertuis' principle. In Section 2 we introduce the observer. In Section 3 we consider applications to some mechanical and hydrodynamical systems. In Section 4 the convergence in the case of positive constant curvature is illustrated by simulations on the sphere.

1 Lagrangian systems on manifolds

Consider the classical mechanical system with n degrees of freedom described by the Lagrangian

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2} g_{ij}(q) \dot{q}^i \dot{q}^j - U(q)$$

where the generalized positions $q \in M$ are written in the local coordinates $(q^i)_{i=1\dots n}$, $g(q) = (g_{ij}(q))_{i=1\dots n, j=1\dots n}$ is a Riemannian metric on the configuration space M , and $U : \mathcal{M} \mapsto \mathbb{R}$ is the potential energy. The Euler-Lagrange equations write in the local coordinates

$$\frac{d}{dt} \left(\frac{\partial}{\partial \dot{q}^i} \mathcal{L} \right) = \frac{\partial}{\partial q^i} \mathcal{L}, \quad i = 1, \dots, n \quad (1)$$

One can prove [1] using $\frac{\partial g^{ik}}{\partial q^l} g_{jk} = -g^{ik} \frac{\partial g_{jk}}{\partial q^l}$ where g^{il} are components of g^{-1} that (1) writes

$$\ddot{q}^i = -\Gamma_{jk}^i(q) \dot{q}^j \dot{q}^k + \frac{\partial}{\partial q^i} U \quad (2)$$

where the Christoffel symbols Γ_{jk}^i are given by $\Gamma_{jk}^i = \frac{1}{2} g^{il} \left(\frac{\partial g_{lk}}{\partial q^j} + \frac{\partial g_{jl}}{\partial q^k} - \frac{\partial g_{jk}}{\partial q^l} \right)$. A curve $\gamma(t)$ which is a critical point of the action

$$S(\gamma) = \int_0^T L(\gamma(t), \dot{\gamma}(t) dt)$$

among all curves with fixed endpoints satisfies the Euler-Lagrange equations (1).

1.1 Lagrangian system in a potential field

Consider a conservative Lagrangian system evolving in an admissible region $\{q \in \mathcal{M} : U(q) < E\}$. The energy of the system $E = T(q, \dot{q}) + U(q) = \frac{1}{2}g_{ij}(q)\dot{q}^i\dot{q}^j + U(q)$ is fixed. According to the Maupertuis principle of least action (see e.g. [5]), in the Riemannian geometry defined by the Jacobi metric $\hat{g}^{ij}(q) = 2(E - U(q))g^{ij}(q)$ and the natural parameter τ such that $\frac{d\tau}{dt} = 2(E - U(q(t)))$, the geodesic flow is a solution of the equation of motion (2). Indeed if the $\hat{\Gamma}_{jk}^i$ are the Christoffel symbols associated to the metric \hat{g} we have

$$\frac{d^2}{d\tau^2}q^i + \hat{\Gamma}_{jk}^i(q)\frac{d}{d\tau}q^j\frac{d}{d\tau}q^k = 0 \quad (3)$$

which writes intrinsically $\hat{\nabla}_{\frac{dq}{d\tau}}\frac{dq}{d\tau} = 0$ and defines the geodesic flow ($\hat{\nabla}$ is the Levi-Civita covariant differentiation of the Jacobi metric).

1.2 Geodesic flow and holonomic constraints

A material particle constrained to lie on a manifold moves along a geodesic [5]. Indeed $E = T$, $\hat{g} = 2Eg$, $d\tau = 2Edt$ ensure the energy T is fixed. According to Maupertuis' principle the motion minimizes $\int_{\gamma} \sqrt{\hat{g}_{ij}\frac{d}{d\tau}q^i\frac{d}{d\tau}q^j}d\tau = (1/\sqrt{2E}) \int_{\gamma} \sqrt{g_{ij}\frac{d}{dt}q^i\frac{d}{dt}q^j}dt$ which is proportional to the geodesic length in the metric g . More generally an inertial motion of a Lagrangian system with k holonomic constraints can be viewed as the inertial motion of a particle constrained to lie on a submanifold of dimension $n - k$ (see e.g. [5] p 90). A conservative Lagrangian system in a potential field with holonomic constraints satisfies the Maupertuis' principle on the configuration submanifold of dimension $n - k$.

2 An intrinsic reduced-observer

Let us build an observer to estimate the velocity \dot{q} of a point q moving along the geodesics of \mathcal{M} with constant speed, when the position q is measured (with noise). First suppose $\mathcal{M} = \mathbb{R}^n$ endowed with Euclidian metric. Let $\dot{q} = v$ and $\dot{v} = 0$. Consider the following reduced-observer

$$\frac{d}{dt}\hat{\xi} = -\frac{\hat{\xi} - q}{\lambda} \quad (4)$$

It can be interpreted as a kind of “elastic” trailer hooked up to q . It allows to estimate the position and speed of the system

$$\xi = q - \lambda v \quad (\text{and thus } \dot{\xi} = v) \quad (5)$$

We want to prove $\hat{\xi} - \xi \rightarrow 0$ and $\hat{v} - v \rightarrow 0$ where $\hat{v} = \frac{d}{dt}\hat{\xi}$. We have $\dot{\hat{v}} = -(1/\lambda)(\hat{v} - v)$. If $\lambda > 0$, $\hat{v} \rightarrow v$ since v is a constant vector. But $\hat{\xi} = q - \lambda \hat{v}$, thus $\hat{\xi} - \xi \rightarrow 0$ and asymptotically $\hat{\xi}$ is moving behind q at distance $\lambda\|v\|$. Now if \mathcal{M} is any Riemannian manifold consider

$$\frac{d}{dt}\hat{\xi} = -\frac{1}{2\lambda} \overrightarrow{\text{grad}}_{\hat{\xi}} D^2(\hat{\xi}, q) \quad \lambda > 0 \quad (6)$$

where $D(\hat{\xi}, q)$ is the geodesic distance between $\hat{\xi}$ and q . The dynamic does not depend on any choice of local coordinates in \mathbb{R}^n , and is a generalization of (4). We want to prove that $D(\hat{\xi}, \xi) \rightarrow 0$ where ξ is a point following q at distance $\lambda\|v\|$ on the geodesic $\{q(t) : t > 0\}$. The parallel transport $\mathcal{T}_{/\hat{\xi} \rightarrow q}$ of $\frac{d}{dt}\hat{\xi}$ to the tangent space at q along the geodesic joining $\hat{\xi}$ and q is an estimation of $v = \dot{q}$. With the following definition, \hat{v} is no more noisy than q

$$\hat{v} = \mathcal{T}_{/\hat{\xi} \rightarrow q} \frac{d}{dt}\hat{\xi} \quad (7)$$

Theorem 1. *Let \mathcal{M} be a Riemannian metric. Let $T < \infty$. Let $t \mapsto q(t) \in \mathcal{M}$ satisfy $\nabla_{\dot{q}}\dot{q} = 0$ for $t \in [0, T]$. Let $\xi(t) = \exp_{q(t)}(-\lambda\|\dot{q}\|)$. Consider the observer (6). Let (8) be the condition*

$$D(\hat{\xi}(t), \xi(t)) \leq e^{-\frac{1}{\lambda}t} D(\hat{\xi}(0), \xi(0)) \quad \forall t \in [0, T] \quad (8)$$

- *Suppose the Riemannian curvature is non-positive in all planes. If for all $t \in [0, T]$, $D(\hat{\xi}(t), q(t))$ is bounded by the injectivity radius $I(t)$ at $q(t)$ (i.e. there exists a unique geodesic joining $\hat{\xi}$ and $q(t)$), (8) is true for all $\lambda > 0$. When the manifold is complete and simply-connected (Hadamard manifold), the injectivity radius is infinite (Cartan-Hadamard theorem) and (8) is always true. In particular $\hat{\xi}(0)$ can be chosen arbitrarily. Moreover*

$$\lambda\|\hat{v}(t) - \dot{q}(t)\| \leq D(\hat{\xi}(t), \xi(t)) \leq e^{-\frac{1}{\lambda}t} D(\hat{\xi}(0), \xi(0)) \quad \forall t \in [0, T] \quad (9)$$

- *Suppose the sectional curvature is bounded from above by $A > 0$. Take $\lambda > \frac{\pi}{4\|v\|_g\sqrt{A}}$. (8) is true as long as the distance $D(\hat{\xi}(t), q(t))$ remains bounded by $\max(\frac{\pi}{4\sqrt{A}}, I(t))$ for all $t \in [0, T]$. If the manifold is simply connected, (8) is true as soon as $D(\hat{\xi}(0), q(0)) < \frac{\pi}{4\sqrt{A}}$.*

Under the assumptions of theorem 1 (second case), if the observation noise amplitude is less than $\frac{\pi}{4\sqrt{A}}$ the observer is also globally convergent in the following sense: take $\hat{\xi}(0)$ equal to the initial position measurement. We have (8) without any information on the initial velocity.

proof The proof consists in two differential geometry lemmas.

Lemma 1. *Let \mathcal{M} be a smooth Riemannian manifold. Let $P \in \mathcal{M}$ be fixed. On the subspace of \mathcal{M} defined by the injectivity radius at P we consider*

$$\frac{d}{dt}x = -\frac{1}{2\lambda} \overrightarrow{\text{grad}_x} D^2(P, x) \quad \lambda > 0 \quad (10)$$

If the sectional curvature is non-positive in all planes, the dynamics is a contraction in the sense of [12], i.e, if δx is a virtual displacement at fixed t we have

$$\frac{d}{dt}\|\delta x\|_g^2 \leq -\frac{2}{\lambda}\|\delta x\|_g^2 \quad (11)$$

where $\|\cdot\|_g$ is the norm associated to the metric g .

If the sectional curvature in all planes is bounded from above by $A > 0$, (11) holds for $D(P, x) < \pi/(4\sqrt{A})$.

proof The virtual displacement is defined [12] as a linear tangent differential form, and can be viewed by duality as a vector of $TM|_x$. Let us define a surface Σ . Let γ_0 be the geodesic joining P to x . A point which is infinitely close to x in the direction δx is linked to P by a geodesic $\gamma_0 + \delta\gamma$. The directions defined by this latter geodesic and γ at P span a 2-plane tangent at P . All the geodesics having a direction tangent to this 2-plane at P span a smooth surface Σ embedded in \mathcal{M} which inherits the Riemannian metric g . Σ is invariant under the flow (10), as the gradient term is tangent to the geodesics heading towards P . We have $x \in \Sigma$ and $\delta x \in T_x\Sigma \subset T_x\mathcal{M}$.

Following [18] (p 177) we use specific coordinates on Σ called “polar coordinates”. Let e_1, e_2 be an euclidian frame of $T_P\Sigma$ for the inherited metric and e_1 be tangent to γ_0 . We define $\Phi : (\sigma, \theta) \mapsto \exp_P(\sigma \cos \theta e_1 + \sigma \sin \theta e_2)$. Σ is parameterized by σ , the geodesic length to P , and θ , the angle in $T_P\Sigma$ with e_1 . In the polar coordinates, the elementary length is given by

$$ds^2 = d\sigma^2 + G(\sigma, \theta)d\theta^2$$

and G satisfies the initial conditions $\sqrt{G} = 0$ and $\frac{\partial \sqrt{G}}{\partial \sigma} = 1$ at $\sigma = 0$. According to a classical result the Gauss curvature at the point (σ, θ) is given

by $K(\sigma, \theta) = \frac{-1}{\sqrt{G(\sigma, \theta)}} \frac{\partial^2 \sqrt{G(\sigma, \theta)}}{\partial \sigma^2}$. We will prove (lemma 2) that the Gaussian curvature at $u = \Phi(\sigma, \theta) \in \Sigma$ is bounded from above by the sectional curvature in the tangent plane to Σ at u : $K(\sigma, \theta) \leq K_{sec}(T_u \Sigma)$. Suppose $K_{sec}(T_u \Sigma) \leq 0$. It implies $K(\sigma, \theta) \leq 0$. Along γ we have

$$\frac{\partial^2 G(\sigma, \theta)}{\partial \sigma^2} = \frac{\partial}{\partial \sigma} \left(2\sqrt{G(\sigma, \theta)} \frac{\partial \sqrt{G(\sigma, \theta)}}{\partial \sigma} \right) = 2 \left(\left(\frac{\partial \sqrt{G(\sigma, \theta)}}{\partial \sigma} \right)^2 - G(\sigma, \theta) K(\sigma, \theta) \right) \geq 0 \quad (12)$$

and thus $\frac{\partial}{\partial \sigma} \left(\sigma \frac{\partial G}{\partial \sigma} \right) \geq \frac{\partial G}{\partial \sigma}$ which yields by integration $\sigma \frac{\partial G}{\partial \sigma} \geq G$ since $G(\sigma, 0) = 0$. In the polar coordinates the dynamics (10) reads

$$\dot{\sigma} = -\frac{1}{\lambda} \sigma; \quad \dot{\theta} = 0$$

Writing $\|\delta x\|^2 = \alpha^2 \delta \sigma^2 + \beta^2 G(\sigma, 0) \delta \theta^2$ we have along the geodesic γ_0 (parameterized by σ and $\theta = 0$) the following inequality, proving (11).

$$\frac{d}{dt} \|\delta x\|^2 = -2 \frac{\alpha^2}{\lambda} \delta \sigma^2 - 2 \beta^2 \frac{\sigma}{\lambda} \frac{\partial G(\sigma, 0)}{\partial \sigma} \delta \theta^2 \leq -\frac{2}{\lambda} \|\delta x\|^2 \quad (13)$$

Suppose now $K_{sec}(T_u \Sigma) \leq A$. Let us find an upper bound on σ under which (11) holds. Let $x(\sigma) = \sqrt{G(\sigma, 0)}$. It suffices to find an upper bound above which (12) is not true, or σ such that $x'(\sigma) = \sqrt{A} x(\sigma)$. We have $x'' = -Kx$, $x(0) = 0$, $x'(1) = 1$. Let $y(\sigma) = \sin(\sqrt{A}\sigma)/\sqrt{A}$. A Taylor expansion in 0 shows that there exists $\alpha > 0$ such that $x(\sigma)/x'(\sigma) < y(\sigma)/y'(\sigma)$ for $0 < \sigma \leq \alpha$. Let $\varphi_K(\sigma)$ be the polar angle of the point $(x'(\sigma), x(\sigma))$ in the phase plane, and $\varphi_A(\sigma)$ the polar angle of $(\cos(\sqrt{A}\sigma), \sin(\sqrt{A}\sigma)/\sqrt{A})$. It is shown in the proof of the Sturm Comparison theorem in [4] that $\varphi_A(\sigma) > \varphi_K(\sigma)$ (as long as $\sigma \leq \pi/\sqrt{A}$). Thus as long as $\sqrt{A}\sigma \leq \arctan 1$ i.e. $\sigma \leq \pi/(4\sqrt{A})$, we have $x'(\sigma) \geq \sqrt{A}x(\sigma)$. Note that we proved at the same time $\pi/(4\sqrt{A})$ is a distance under which the exponential is an injection. If the manifold is simply connected, the exponential is a surjection and $\pi/(4\sqrt{A})$ is less than the injectivity radius.

Lemma 2. *Let \mathcal{M} be a smooth manifold. Let $P \in \mathcal{M}$. Let E be a two-dimensional vectorial space of $T_P \mathcal{M}$. Let ω be a neighborhood of P in E such that the restriction of the exponential map σ to ω is a diffeomorphism in \mathcal{M} . $\Sigma = \sigma(\omega)$ is submanifold of dimension 2. Its Gaussian curvature at any $u \in \Sigma$ is bounded from above by the sectional curvature in the tangent plane to Σ at u :*

$$K(u) \leq K_{sec}(T_u \Sigma)$$

proof The proof is written after computations of Ivan Kupka. Let Σ be the surface constructed at lemma 1 and Φ the map associated to the polar coordinates. The metric inherited on Σ writes

$$ds^2 = \left\| \frac{\partial \Phi}{\partial \sigma} dr + \frac{\partial \Phi}{\partial \theta} d\theta \right\|_g^2 = \left\| \frac{\partial \Phi}{\partial \sigma} \right\|_g^2 dr^2 + 2 \left\langle \frac{\partial \Phi}{\partial \sigma}, \frac{\partial \Phi}{\partial \theta} \right\rangle_g d\sigma d\theta + \left\| \frac{\partial \Phi}{\partial \theta} \right\|_g^2 d\theta^2$$

For fixed θ the curve $\gamma_\theta : \sigma \mapsto \Phi(\sigma, \theta)$ is a geodesic and thus $\left\| \frac{\partial \Phi}{\partial \sigma} \right\|_g^2 = 1$. Let $J(\sigma, \theta) = \frac{\partial \Phi}{\partial \theta}$. For fixed θ , $J_\theta : \sigma \mapsto J(\sigma, \theta)$ is a Jacobi field along γ_θ . Moreover $J(0, \theta) = -\sin \theta e_1 + \cos \theta e_2$ is orthogonal to this geodesic in P . It is well known (Jacobi field properties) that it implies J is orthogonal to γ_θ at any point. Thus $\left\langle \frac{\partial \Phi}{\partial \sigma}, \frac{\partial \Phi}{\partial \theta} \right\rangle_g = 0$ and $ds^2 = d\sigma^2 + \|J(\sigma, \theta)\|_g^2 d\theta^2$, and the Gaussian curvature is given by $K(\sigma, \theta) = \frac{-1}{\|J(\sigma, \theta)\|_g} \frac{\partial^2 \|J(\sigma, \theta)\|_g}{\partial \sigma^2}$ (see [18]). Consider the Levi-Civita covariant differentiation ∇ of the metric g . We have $\frac{\partial \|J(\sigma, \theta)\|_g}{\partial \sigma} = \frac{\langle \nabla_\sigma J(\sigma, \theta), J(\sigma, \theta) \rangle}{\|J(\sigma, \theta)\|_g}$ and

$$\begin{aligned} \frac{\partial^2 \|J(\sigma, \theta)\|_g}{\partial \sigma^2} &= \frac{\langle \nabla_\sigma^2 J(\sigma, \theta), J(\sigma, \theta) \rangle}{\|J(\sigma, \theta)\|_g} + \frac{\langle \nabla_\sigma J(\sigma, \theta), \nabla_\sigma J(\sigma, \theta) \rangle}{\|J(\sigma, \theta)\|_g} \\ &\quad - \frac{\langle \nabla_\sigma J(\sigma, \theta), J(\sigma, \theta) \rangle^2}{\|J(\sigma, \theta)\|_g^3} \end{aligned}$$

According to the Jacobi equation we have $\nabla_\sigma^2 J(\sigma, \theta) + R(J(\sigma, \theta), \frac{\partial \Phi(\sigma, \theta)}{\partial \sigma}) \frac{\partial \Phi(\sigma, \theta)}{\partial \sigma} = 0$. Thus the Gaussian curvature of Σ satisfies

$$K(\sigma, \theta) = K_{sec}(T_u \Sigma) + \frac{\langle \nabla_\sigma J(\sigma, \theta), J(\sigma, \theta) \rangle^2 - \|J(\sigma, \theta)\|_g^2 \|\nabla_\sigma J(\sigma, \theta)\|_g^2}{\|J(\sigma, \theta)\|_g^4}$$

where $K_{sec}(T_u \Sigma) = \frac{\langle R(J(\sigma, \theta), \frac{\partial \Phi(\sigma, \theta)}{\partial \sigma}) \frac{\partial \Phi(\sigma, \theta)}{\partial \sigma}, J(\sigma, \theta) \rangle}{\|J(\sigma, \theta)\|_g^2}$ is the sectional curvature at $u = \Phi(\sigma, \theta)$, and where we used that $J(\sigma, \theta)$ is orthogonal to $\frac{\partial \Phi(\sigma, \theta)}{\partial \sigma}$. Cauchy-Schwarz implies that the fraction above is negative and $K(\sigma, \theta) \leq K_{sec}(T_u \Sigma)$.

We can now prove theorem 1. Suppose the sectional curvature is nonpositive in all planes. Using the contraction [12] interpretation in the appendix of [3] we see that if $\hat{\xi}_1, \hat{\xi}_2$ are solutions of (6) which remain at a distance of q bounded by the injectivity radius for $0 \leq t \leq T$ we have

$$D(\hat{\xi}_1(t), \hat{\xi}_2(t)) \leq e^{-\frac{1}{\lambda} t} D(\hat{\xi}_1(0), \hat{\xi}_2(0)) \quad \forall t \in [0, T]$$

since by lemma 1 (6) is a contraction. The system “forgets” its initial condition. So (8) holds if $\xi(t)$ is a solution of (6). This is true since $0 = \frac{d}{dt} D(\xi, q) = \|v\|_g - \|\dot{\xi}\|_g = \|v\|_g - \frac{1}{\lambda} D(\xi, q)$. To prove (9) notice $\frac{\partial \sqrt{G(\sigma, \theta)}}{\partial \sigma} > 0$ implies $\int_0^\alpha G(\sigma, \theta) d\theta \geq \int_0^\alpha \sigma^2 d\theta$. Thus the angle α between



Figure 1: Left: Geodesic deviation on a manifold of negative curvature. (10) writes in polar coordinates (σ, θ) : $\dot{\sigma} = -\frac{\sigma}{\lambda}$; $\dot{\theta} = 0$. The distance $\|\delta x\|$ between neighbors x_1 and x_2 decreases at a rate at least $\frac{1}{\lambda}$. Right: Geodesic deviation on the sphere.

\hat{v} and v in $T_q\mathcal{M}$ is smaller than the angle β corresponding to the Euclidian case $K \equiv 0$, $G(\sigma, \theta) = \sigma^2$. In this latter case $q, \xi, \hat{\xi}$ is an Euclidian triangle and thus $\|\hat{v} - \hat{q}\| \leq D(\hat{\xi}, \xi)/\lambda$ (make a drawing to see the homothety).

Suppose the sectional curvature is bounded from above by A and $\lambda > \frac{\pi}{4\|v\|_g\sqrt{A}}$. Then if $D(\hat{\xi}(0), q(0)) < \frac{\pi}{4\sqrt{A}}$ we have $D(\hat{\xi}(t), q(t)) < \frac{\pi}{4\sqrt{A}}$ for all $t > 0$. Indeed $\frac{d}{dt}D(\hat{\xi}, q) \leq \langle \text{grad}_q D(\xi, q), v \rangle - \|\frac{d}{dt}\hat{\xi}\|_g \leq \|v\|_g - \frac{1}{\lambda}D(\hat{\xi}, q)$. Thus $D(\hat{\xi}, q) = \frac{\pi}{4\sqrt{A}}$ implies $\frac{d}{dt}D(\hat{\xi}, q) < 0$. By lemma 1 (6) is a contraction. (8) holds by the same token as last paragraph. When the curvature is constant, it is easy to prove that $\|\hat{v} - \hat{q}\| \rightarrow 0$ exponentially at rate $1/\lambda$ since $G(\sigma, \theta) = \frac{\sin \sqrt{K}\sigma}{\sqrt{K}}$. Intuitively we suspect it remains true if the curvature is not varying too fast along the geodesics but we have no proof for this claim. Nevertheless for fixed q, ξ and time t , if $D(\hat{\xi}, \xi) < \pi/(4\sqrt{A})$ we have in the limit $\|\hat{v} - \hat{q}\| \rightarrow 0$ as $D(\hat{\xi}, \xi) \rightarrow 0$.

3 Applications

3.1 Lagrangian mechanical system

Proposition 1. *Consider any Lagrangian system in a potential field in the admissible region defined by $U < E$. The observer*

$$\frac{d}{dt}\hat{\xi} = -\frac{1}{\lambda}(E - U(q)) \overrightarrow{\text{grad}}_{\hat{\xi}} D_{\hat{g}}^2(\hat{\xi}, q) \quad (14)$$

is such that the theorem 1 is valid in the Maupertuis time and Jacobi metric.

proof To prove this proposition, one can apply the Maupertuis' principle (see section 1.1). In Maupertuis' time $\tau = \int_0^t 2(E - U(q(t)))dt$ the motion is a geodesic flow on the configuration space with modified metric \hat{g} , with $\|v\|_{\hat{g}} =$

1 as $\hat{g}_{ij}(q) \frac{dq^i}{d\tau} \frac{dq^j}{d\tau} = 2(E - U)g_{ij}(q) \frac{dq^i}{dt} \frac{dq^j}{dt} (\frac{dt}{d\tau})^2 = 1$. The observer defined by $\frac{d}{d\tau} \hat{\xi} = -\frac{1}{2\lambda} \overrightarrow{\text{grad}}_{\hat{\xi}} D_{\hat{g}}^2(\hat{\xi}, q)$, $\lambda > 0$ where $D_{\hat{g}}$ is the distance associated to Jacobi metric, is such that $\hat{v} = \mathcal{T}_{\hat{\xi} \rightarrow q} \frac{d}{d\tau} \hat{\xi}$ is an estimation of $\frac{d}{d\tau} q$.

For instance, R. Montgomery studied in a recent paper [15] the Newtonian equal-mass three bodies problem, with zero momentum and when the potential is taken equal to $1/r^2$: the Jacobi metric has negative curvature everywhere (except at two points). The reduced observer (14) is thus globally convergent for a three bodies system which is sensible to initial conditions.

Remark 1. *For a conservative system, the total energy E , or equivalently the initial kinetic energy needs to be known to compute the Jacobi metric. However no information about the initial direction of the velocity is required. In the particular case of an inertial motion, the metric g is used and no information on the initial velocity needs to be known (except an upper bound to tune the gains in the case of positive curvature).*

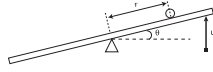


Figure 2: Ball and Beam

Remark 2. *Let us consider now a non-conservative system: the ball and beam of [3] with a torque control u (see fig 2). The observer [3] is only locally convergent. Observer (6) can be used complementarily to provide a globally convergent estimator with the following little experiment. Set $\hat{\xi}(0) = q(0)$. Maintain $u \equiv 0$ (no control). The characteristic time of convergence of the observer (6) is $\tau_0 = \lambda$ in the Maupertuis time. After a few τ_0 the observer (6) provides the observer of Aghannan and Rouchon an initial estimation of the velocity close to the true one and from that moment u can vary freely again: [3] converges. There are two restrictions: the initial energy must be approximately known, and the speed must be bounded from below during the experiment.*

3.2 Motion of a perfect incompressible fluid

The goal of this section is to show that the reduced observer could possibly be applied to more complicated systems. No formal proof is given but only heuristic discussions. The observer could be used in velocimetry as a (soft) velocimeter when the flow is seeded with particles that can be observed and is modeled by Euler equations.

3.2.1 A reduced observer

Let us first introduce some results and notations of [5, 6, 17]. Let Ω be a domain of \mathbb{R}^3 bounded by a surface $\delta\Omega$. Let \vec{v} be the velocity field of an ideal incompressible perfect fluid with density ρ which fills the domain Ω . The motion is described by the Euler equation

$$\frac{d}{dt}\vec{v} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p \quad (15)$$

where p is the pressure. Let $\text{SDiff } \Omega$ be the Lie group of all diffeomorphisms that preserve the Euclidian volume. Its Lie algebra \mathcal{U} is the set of all vector fields of Ω of null divergence, and tangent to $\delta\Omega$. Consider the scalar product on the Lie algebra

$$\forall \vec{v}, \vec{w} \in \mathcal{U}, \quad \langle \vec{v}, \vec{w} \rangle = \rho \int \int \int_{\Omega} \vec{v}(x) \cdot \vec{w}(x) dx \quad (16)$$

Let $\vec{v}(t) \in \mathcal{U}$ be a solution of (15). Let $\phi_t^{\vec{v}}(x) = z(t)$ be the position of every fluid particle after a time t , obtained by integration on $[0, t]$ of the system $\frac{d}{ds}z = \vec{v}(s, z)$, $z(0) = x$. $\phi_t^{\vec{v}}$ is a diffeomorphism for any $t > 0$, and the motion of the fluid is described by a curve $t \mapsto \phi_t^{\vec{v}}$ on $\text{SDiff } \Omega$. Suppose t is fixed. After a small time τ the diffeomorphism describing the fluid will be $\exp_{Id}(\tau \vec{v}(t))\phi_t^{\vec{v}}$ up to second order terms in τ . It implies $\vec{v}(t) = DR_{(\phi_t^{\vec{v}})^{-1}} \frac{d}{dt}\phi_t^{\vec{v}}$ where DR_g denotes the tangent map induced by right multiplication by g on the group. Thus the kinetic energy of the fluid $T = \frac{1}{2} \langle \vec{v}, \vec{v} \rangle$ defines a right-invariant metric. The least action principle implies that the fluid motion $t \mapsto \phi_t^{\vec{v}}$ is a geodesic flow on $\text{SDiff } \Omega$ endowed with the kinetic energy metric. Thus $\nabla_{\vec{v}}^{LC} \vec{v} = 0$ where the Levi-Civita covariant differentiation ∇^{LC} is given by $\nabla_{\vec{v}}^{LC} \vec{\eta} = \frac{\partial}{\partial t} \xi + (\vec{v} \cdot \nabla) \vec{\eta} + \nabla \alpha$ and α is a real function such that $\nabla_{\vec{v}}^{LC} \vec{\eta} \in \mathcal{U}$. For fixed t , define the virtual displacement $\delta\phi(s, \cdot)$ corresponding to δx in lemma 1:

$$\delta\phi(s, x) \approx \phi_s^{\vec{v} + \delta\vec{v}}(x) - \phi_s^{\vec{v}}(x)$$

[17] proves it can be identified to an element of \mathcal{U} and it satisfies the Jacobi equation $(\nabla_{\vec{v}}^{LC})^2 \delta\phi + A_{\vec{v}}(\delta\phi) = 0$ where $\langle A_{\vec{v}}(\vec{\eta}), \vec{\eta} \rangle$ is the curvature of the plane section spanned the orthonormal frame $\vec{v}, \vec{\eta}$.

The reduced observer is defined intrinsically and can formally be applied to this fluid velocity estimation problem. The theorem 1 is valid, as the proof is only made of intrinsic calculations, and its core is the Jacobi equation which gives conditions under which $\sigma \frac{\partial}{\partial \sigma} G \geq G$. The observer's state $\hat{\xi}$ is a virtual fluid defined as a solution of (6) where q is replaced by $\phi_t^{\vec{v}}$. Using the right invariance of the metric, one can consider $\hat{\zeta}(t) = \hat{\xi}(t) \circ (\phi_t^{\vec{v}})^{-1}$ at any $t > 0$,

and the update is given by $\hat{v} = -DR_{\phi_t^{\vec{v}}} \vec{w} \in T_{\phi_t^{\vec{v}}} \text{SDiff } \Omega$ where $\hat{\zeta} = \phi_1^{\vec{w}}$. Thus only the inverse of the exponential map $\phi_1^{\vec{w}}$ is required. Note that $\hat{\zeta}$ must remain in the identity connected component.

3.2.2 Discussion on the convergence and curvature

When the curvature is bounded from above by $-B^2 < 0$, the geodesic flow is sensitive to initial conditions, and amits ergodic properties [5]. Surprisingly, in this case the observer is globally exponentially convergent by theorem 1. When there are always sections with negative curvature along a geodesic, it is commonly assumed that the sensity to initial conditions is still valid.

We have the following formal convergence result: consider a sinusoidal parrallel stationary motion of a fluid in the tore $T^2 = \{(x, y), x \bmod 2\pi, y \bmod 2\pi\}$ given by the current function $\psi = \cos(kx + ly)$ with $k, l \in \mathbb{N}$, and the velocity vector field $\vec{v} = \text{rot } \psi$. Take $\hat{\xi}(0) = \phi_0^{\vec{v}}$ for (6). Then $\|\hat{v}(t) - \vec{v}\|$ converges exponentially to 0. The proof is obvious as both points belong to the same geodesic. But one can expect a great robustness to measurement noise. Indeed [5] proves the motion defined by ψ is a geodesic of $\text{SDiff } T^2$, and the curvature is non positive in *all* planes containing $\vec{v}(x, y)$. Moreover it is zero only in a family of planes of null measure. But by theorem 1 negative curvature implies global stability, and small positive curvature implies a large basin of attraction.

More generally, Arnol'd [5] considers the group $\text{S}_0\text{Diff } T^2$ of diffeomorphisms preserving the center of gravity. Calculations show the curvature is positive “only in a few sections”. He suggests, as a concluding remark of appendix 2, to consider the mean curvature along paths to characterize the stability of the flow. As a consequence, if the atmosphere was a bidimensional incompressible fluid on the earth viewed as T^2 (identify opposite sides of the planisphere), the wind should be known up to 5 decimals for a two-months weather's prediction. Following this suggestion, as the curvature is positive in only in a few sections, one could expect a good global behavior of the observer.

Remark 3. *One could object that the problem is as simple as identifying a geodesic from position measurements. But trying to fit the geodesic by a least-square type method (for instance) would be less robust to perturbations: a slight ponctual change in the velocity angle at some point leads to large changes in the trajectory. However the observer follows $\phi_t^{\vec{v}}$. Moreover the observer is intrinsic, its gain is easy to tune, and no differentiation is required.*

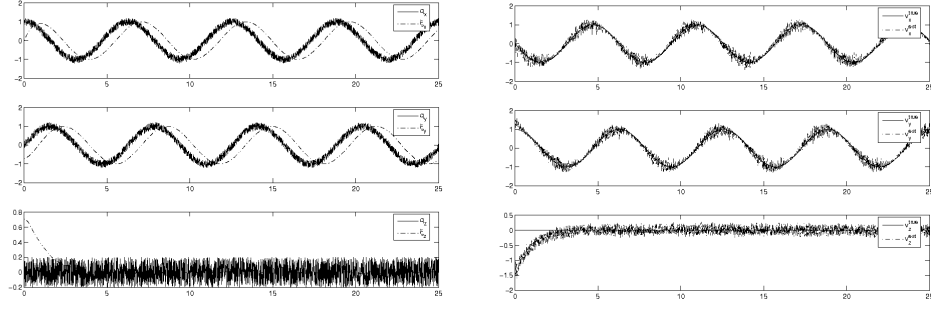


Figure 3: Simulations on the sphere for $\lambda = \pi/4$. Left: q (plain line) et $\hat{\xi}$ (dashed line). Right: velocity v (plain line) and estimation \hat{v} (dashed line)

4 Simulations on the sphere

Consider the inertial motion of a material point constrained to lie on the sphere \mathbb{S}^2 . The speed is constant (see section 1.2) and assumed to be equal to 1. One can always choose coordinates $q = (q_x, q_y, q_z) \in \mathbb{R}^3$ such that the motion writes: $\dot{q}_x(t) = \cos(t)$, $\dot{q}_y(t) = \sin(t)$, $\dot{q}_z(t) = 0$. Let $\hat{\xi} = (\hat{\xi}_x, \hat{\xi}_y, \hat{\xi}_z) \in \mathbb{R}^3$. The observer equation (6) writes

$$\begin{aligned} \frac{d}{dt}\hat{\xi}_x &= \frac{1}{\lambda}\varphi \frac{((q \wedge \hat{\xi}) \wedge \hat{\xi})_x}{\|(q \wedge \hat{\xi}) \wedge \hat{\xi}\|}, \\ \frac{d}{dt}\hat{\xi}_y &= \frac{1}{\lambda}\varphi \frac{((q \wedge \hat{\xi}) \wedge \hat{\xi})_y}{\|(q \wedge \hat{\xi}) \wedge \hat{\xi}\|}, \\ \frac{d}{dt}\hat{\xi}_z &= \frac{1}{\lambda}\varphi \frac{((q \wedge \hat{\xi}) \wedge \hat{\xi})_z}{\|(q \wedge \hat{\xi}) \wedge \hat{\xi}\|} \end{aligned}$$

where $\lambda < 0$ and φ is the angle between q and $\hat{\xi}$. As the geodesics of the sphere are great circles, φ is the geodesic length between those two points. The initial conditions are : $q(0) = [1, 0, 0]^T$ and $\hat{\xi}(0) = \frac{1}{\sqrt{2}}[0, 1, 1]^T$. To simulate the sensor's imperfections a white noise whose amplitude is 20% of the maximal value of the signal was added. $\hat{\xi}$ converges to the equator, and asymptotically follows q at a distance $|\lambda|$. The parallel transport \hat{v} of $\frac{d}{dt}\hat{\xi}$ is an estimation of v . In fact for $\lambda < \pi/2$ the observer always converges in simulation, and for $\lambda > \pi/2$ it does not (see fig 1).

5 Conclusion

We designed a nonlinear globally convergent reduced observer for conservative Lagrangian systems. The observer is intrinsic and converges despite the effects of curvature: instability of the flow and gyroscopic terms. The tuning of the gains is simple. The only gain is a scalar to be chosen in function of the noise and the maximal curvature. It can be used for velocity estimation for all systems described by geodesic flows ($\nabla_v v = 0$), notably conservative Lagrangian system, and the motion of an incompressible fluid. Using the Maupertuis principle this work could be extended to the case of a mixture of compressible fluids [16].

Unfortunately when the motion is described by $\nabla_v v = S(q)$ with S known (Lagrangian system with external forces) the reduced observer does not converge. Including such terms S remains an open question. As a concluding remark, note that the article gives insight in the link between convergence and geometrical structure of the model in the theory of observers (complementing the work of [3] and more recent results [8]). The observer's principle can be viewed as the simplest pursuit algorithm on manifolds (proportional feedback). A quantitative link between the effectiveness of the pursuit and the topography of the environment (Riemannian curvature) has been established.

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References

- [1] R. Abraham and J.E. Marsden. *Foundations of Mechanics*. Addison-Wesley (updated 1985 printing), second edition, 1985.
- [2] N. Aghannan and P. Rouchon. On invariant asymptotic observers. In *Proceedings of the 41st IEEE Conference on Decision and Control*, volume 2, pages 1479– 1484, 2002.
- [3] N. Aghannan and P. Rouchon. An intrinsic observer for a class of lagrangian systems. *IEEE Trans. Automat. Control*, 48(6):936–945, 2003.
- [4] V. Arnold. *Ordinary Differential Equations*. Mir Moscou, 1974.

- [5] V. Arnold. *Mathematical Methods of Classical Mechanics*. Mir Moscou, 1976.
- [6] V. Arnol'd. Sur la géométrie différentielle des groupes de lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier*, 16:319–361, 1966.
- [7] S. Bonnabel, P. Martin, and P. Rouchon. Symmetry-preserving observers. *IEEE Trans. Automat. Control*, volume 53, 2008.
- [8] S. Bonnabel, Ph. Martin, and P. Rouchon. Non-linear symmetry-preserving observers on lie groups. *Accepted for publication in IEEE-TAC*, <http://arxiv.org/abs/0707.2286>, 2008.
- [9] S. Bonnabel, M. Mirrahimi, and P. Rouchon. Observer-based hamiltonian identification for quantum systems. *Accepted for publication, Automatica*, 2008.
- [10] F. Bullo, N.E. Leonard, and A.D. Lewis. Controllability and motion algorithms for underactuated lagrangian systems on lie groups. *IEEE Trans. Automat. Control*, 35:1437–1454, 2000.
- [11] A.D. Lewis and R.M. Murray. Configuration controllability of simple mechanical control systems. *SIAM J. Control Optim.*, 35:766–790, 1997.
- [12] W. Lohmiller and J.J.E. Slotine. On metric analysis and observers for nonlinear systems. *Automatica*, 34(6):683–696, 1998.
- [13] R. Mahony, T. Hamel, and J-M Pfimlin. Non-linear complementary filters on the special orthogonal group. *Accepted for publication in IEEE-AC*.
- [14] D. H. S. Maithripala, W. P. Dayawansa, and J. M. BERG. Intrinsic observer-based stabilization for simple mechanical systems on lie groups. *SIAM J. Control and Optim.*, 44:1691–1711, 2005.
- [15] R. Montgomery. Hyperbolic pants fit a three-body problem. *Ergod. Thy. Dyn. Sys.*, 25.
- [16] P. Rouchon. Dynamique des fluides parfaits, principe de moindre action, stabilité lagrangienne. Technical Report 13/3446 EN, ONERA, september 1991.

- [17] P. Rouchon. On the Arnol'd stability criterion for steady-state flows of an ideal fluid. *European Journal of Mechanics /B Fluids*, 10:651–661, 1991.
- [18] J.J. Stoker. *Differential Geometry*. Wiley-Interscience, 1969.